Divisibility Rules Justified
Grades 6-8

Standards: 7 MR3.3 Develop generalizations of the results obtained and the strategies used and apply them to new problem situations.
7.EE.1 Use properties of operations to generate equivalent expressions. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

Materials: Class set of properties cards (pages 7-11) for students to use in pairs (or small groups)
Partial notes handout (pages 12-13), one per student

Introduction: "Divisibility rules can be quick shortcuts for factoring multi-digit numbers. Today we are going to examine why they work using proofs with variables. In middle school, you are not expected to be able to write proofs like this, but you will be able to justify each step, working with your partner (or small group)."

Ex 1 (I Do): Divisibility rule for 3
A number is divisible by 3 *iff* (if and only if) the sum of its digits is a multiple of 3.

Proof:
Let N be a four digit number such that $N = abcd$.

The thousands digit times 1000, the hundreds digit times 100, the tens digit times 10 plus the ones digit.

$N = a \bullet 1000 + b \bullet 100 + c \bullet 10 + d$ Expanded notation

$N = a(999+1) + b(99+1) + c(9+1) + d$ Decomposition

$N = 999a + a + 99b + b + 9c + c + d$ Distributive Property of Multiplication Over Addition

$N = 999a + 99b + 9c + a + b + c + d$ Commutative Property of Addition

$N = 3 \bullet 333a + 3 \bullet 33b + 3 \bullet 3c + (a+b+c+d)$ Decomposition

In this step, we decompose into factors to show parts that are divisible by 3.

Because it has 3 as a factor, $3(333a) + 3(33b) + 3(3c)$ is divisible by 3.
$\therefore N$ would only be divisible by 3 if $a+b+c+d$, the remainder, is divisible by 3.
And, if $a+b+c+d$ is divisible by 3, then N is divisible by 3.

Using the Shortcut With Numbers:

Is 9831 divisible by 3?

$9+8+3+1$
$=17+3+1$
$=21$

21 is divisible by 3, so we know that 9831 is divisible by 3.

Is 1802 divisible by 3?

$1+8+0+2$
$=9+2$
$=11$

11 is not divisible by 3, so we know that 1802 is not divisible by 3.

Ex 2 (We Do): [Hand out properties cards to pairs or groups.]

Divisibility rule for 4
A number is divisible by 4 *iff* the last two digits form a number that is a multiple of 4.

Proof:
Let N be a four digit number such that $N = abcd$.

Think-Pair-Share "Think about what it is called when we take a number apart into factors? Check with your partner and choose a card. Show me your cards."

$N = a \bullet 1000 + b \bullet 100 + c \bullet 10 + d$ Expanded notation

$N = a \bullet 250 \bullet 4 + b \bullet 25 \bullet 4 + c \bullet 10 + d$ Decomposition

$N = 4 \bullet 250a + 4 \bullet 25b + c \bullet 10 + d$ Commutative Property of Multiplication

TPS "Think about which property tells us we can change the order of factors? Check with your partner and choose a card. Show me your cards." Note that multiplication can be written with parentheses or a dot.

Because it has 4 as a factor, $4(250a) + 4(25b)$ is divisible by 4.
In other words, ALL hundreds and thousands are divisible by 4.
$\therefore N$ would only be divisible by 4 if $10c + d$, the number formed by the last two digits, is divisible by 4.
And, if $10c + d$ is divisible by 4, then N is divisible by 4.

TPS "If all hundreds and thousands are divisible by 4, what about ten-thousands? "

Using the Shortcut With Numbers:

Is 7812 divisible by 4?

The last two digits form the number 12, which is a multiple of 4. Since 12 is divisible by 4, we know that 7812 is divisible by 4.

Is 1831 divisible by 4?

The last two digits form the number 31, which is not a multiple of 4. Since 31 is not divisible by 4, we know that 1831 is not divisible by 4.

Ex 3 (You Do Together): Think-Pair-Share to justify each step.

Divisibility rule for 9

A number is divisible by 9 *iff* (if and only if) the sum of its digits is a multiple of 9.

Proof:
Let N be a four digit number such that $N = abcd$.

TPS "What lets us write each variable by its place value? Check with your partner. Show me your cards."

$N = a \bullet 1000 + b \bullet 100 + c \bullet 10 + d$ Expanded notation

TPS "Why can we break apart 1000 into 999 plus 1? Check with your partner. Show me your cards."

$N = a(999+1) + b(99+1) + c(9+1) + d$ Decomposition

$N = 999a + a + 99b + b + 9c + c + d$ Distributive Property of Multiplication Over Addition

TPS "We just multiplied a by every term in the parentheses. Justify that step. Check with your partner. Show me your cards."

$N = 999a + 99b + 9c + a + b + c + d$ Commutative Property of Addition

TPS "What lets us change the order of these terms? Check with your partner. Show me your cards."

$N = 9 \bullet 111a + 9 \bullet 11b + 9 \bullet 1c + (a + b + c + d)$ Decomposition

TPS "Why can we break apart 999 into 9 times 111? Check with your partner. Show me your cards."

Because it has 9 as a factor, $9(111a) + 9(11b) + 9(1c)$ is divisible by 9.
$\therefore N$ would only be divisible by 9 if $a+b+c+d$, the remainder, is divisible by 9.
And, if $a+b+c+d$ is divisible by 9, then N is divisible by 9.

You Do: Using the Shortcut With Numbers

Is 9831 divisible by 9?

$9+8+3+1$
$=17+3+1$
$=21$

21 is not divisible by 9, so we know that 9831 is not divisible by 9.

Is 2574 divisible by 9?

$2+5+7+4$
$=7+11$
$=18$

18 is divisible by 9, so we know that 2574 is divisible by 9.

Discussion question: "What do the divisibility rules for 3 and 9 have in common? " [We use the sum of the digits to check for divisibility.]

Ex 4 (You Do Together):

Divisibility rule for 8
A number is divisible by 8 *iff* the last three digits form a number that is a multiple of 8.

Proof:
Let N be a four digit number such that $N = abcd$.

TPS "What lets us write each variable by its place value? Check with your partner. Show me your cards."

$N = a \bullet 1000 + b \bullet 100 + c \bullet 10 + d$ Expanded notation

TPS "Think about what it is called when we take a number apart into factors? Check with your partner. Show me your cards."

$N = a \bullet 125 \bullet 8 + b \bullet 100 + c \bullet 10 + d$ Decomposition

$N = 8 \bullet 125a + b \bullet 100 + c \bullet 10 + d$ Commutative Property of Multiplication

TPS "Think about which property tells us we can change the order of factors? Check with your partner. Show me your cards." Note that multiplication can be written with parentheses or a dot.

Because it has 8 as a factor, $8(125a)$ is divisible by 8.
In other words, ALL thousands are divisible by 8.
$\therefore N$ is only divisible by 8 if $100b + 10c + d$, the number formed by the last three digits, is divisible by 8.
And, if $100b + 10c + d$ is divisible by 8, then N is divisible by 8.

TPS "If all thousands are divisible by 8, what about ten-thousands? "

You Do: Using the Shortcut With Numbers

Is 7812 divisible by 8?

The last three digits form the number 812. 800 is divisible by 8, but 12 is not. Since 812 is not divisible by 8, we know that 7812 is not divisible by 8.

Is 2320 divisible by 8?

The last three digits form the number 320, which is $40 \bullet 8$. Since 320 is divisible by 8, we know that 2320 is divisible by 8.

Discussion question: "What are the similarities and differences between the divisibility rules for 4 and 8?" [We only have to check the last digits, but check 2 digits for 4 and 3 digits for 8.]

Warm-Up: Divisibility Rules

CST/CAHSEE: 7MR 3.3

The winning number in a contest was less than 50. It was a multiple of 3, 5 and 6. What was the number?

A 14

B 15

C 30

D It cannot be determined.

Review: 6AF 1.3/6.EE.4

Simplify:

a) $2 \bullet 7 \bullet 3$

b) $3 \bullet 2 \bullet 7$

Why are the expressions above equivalent?

Current: 7AF 1.4/6.EE.2b

How many terms are in the expression below? How do you know?

$$3y^3 - 5y^2 + 2y + 11$$

Other: 6AF 1.3/6.EE.3

Which equation shows the distributive property?

A $(5+y)+2=2+(5+y)$

B $2+(5+y)=(2+5)+y$

C $(5+y)+0=(5+y)$

D $5(y+2)=5y+10$

Warm-Up Answer Key

y

CST/CAHSEE: 7MR 3.3

The winning number in a contest was less than 50. It was a multiple of 3, 5 and 6. What was the number?

A 14 [*Student added 3, 5 and 6.*]

B 15 [*Multiple of 3 and 5*]

C 30

D It cannot be determined.

6AF 1.3/6.EE.4

Simplify:

a) $2 \bullet 7 \bullet 3 = 42$

b) $3 \bullet 2 \bullet 7 = 42$

Why are the expressions above equivalent?

Commutative Property of Multiplication

x

Current: 7AF 1.4/6.EE.2b

How many terms are in the expression below? How do you know?

$$3y^3 - 5y^2 + 2y + 11$$

There are 4 terms because there are 4 parts being added or subtracted.

Other: 6AF 1.3/6.EE.3

Which equation shows the distributive property?

A $(5+y)+2 = 2+(5+y)$

Commutative Property of Addition

B $2+(5+y) = (2+5)+y$

Associative Property of Addition

C $(5+y)+0 = (5+y)$

Additive Identity

D $5(y+2) = 5y+10$

Decomposition

Breaks numbers apart

Terms $1000 = 999 + 1$

Factors $1000 = 8 \bullet 125$

$999 = 3 \bullet 333$

Distributive Property of Multiplication Over Addition

The factor next to the parentheses is multiplied by every term in that quantity.

$$a(999+1)=999a+1a$$

$$3(x+5)=3x+15$$

$$(x+5)3=3x+15$$

Commutative Property of Addition

Changes order of terms

$$999a+a+99b+b+9c+c+d=999a+99b+9c+a+b+c+d$$

$$15+7=7+15$$

$$(5+y)+2=2+(5+y)$$

Commutative Property of Multiplication

Changes order of factors

$$2 \bullet 7 \bullet 3 = 7 \bullet 3 \bullet 2$$

$$a \bullet 125 \bullet 8 = 8 \bullet 125 \bullet a$$

Note: Multiplication can be written with parentheses or a dot.

Expanded Notation

Rewriting a number by place value

$$8649 = 8 \bullet 1000 + 6 \bullet 100 + 4 \bullet 10 + 9$$

Name____ Period ____

Notes for Divisibility Rules

Ex 1) Divisibility rule for 3: A number is divisible by 3 *iff* (if and only if) the sum of its digits is a multiple of 3.

Proof:

Let N be a four digit number such that $N = abcd$.

$N = a \bullet 1000 + b \bullet 100 + c \bullet 10 + d$ ______________________________

$N = a(999+1) + b(99+1) + c(9+1) + d$ ______________________________

$N = 999a + a + 99b + b + 9c + c + d$ ______________________________

$N = 999a + 99b + 9c + a + b + c + d$ ______________________________

$N = 3 \bullet 333a + 3 \bullet 33b + 3 \bullet 3c + (a + b + c + d)$ ______________________________

Because it has 3 as a factor, $3(333a) + 3(33b) + 3(3c)$ is divisible by 3 .

$\therefore N$ would only be divisible by 3 if $a+b+c+d$, the remainder, is divisible by 3 .

And, if $a+b+c+d$ is divisible by 3 , then N is divisible by 3 .

Ex 2) Divisibility rule for 4: A number is divisible by 4 *iff* the last two digits form a number that is a multiple of 4.

Proof:

Let N be a four digit number such that $N = abcd$.

$N = a \bullet 1000 + b \bullet 100 + c \bullet 10 + d$ ______________________________

$N = a \bullet 250 \bullet 4 + b \bullet 25 \bullet 4 + c \bullet 10 + d$ ______________________________

$N = 4 \bullet 250a + 4 \bullet 25b + c \bullet 10 + d$ ______________________________

Because it has 4 as a factor, $4(250a) + 4(25b)$ is divisible by 4 .

In other words, ALL hundreds and thousands are divisible by 4 .

$\therefore N$ would only be divisible by 4 if $10c + d$, the number formed by the last two digits, is divisible by 4 .

And, if $10c + d$ is divisible by 4 , then N is divisible by 4 .

Ex 3) Divisibility rule for 9: A number is divisible by 9 *iff* (if and only if) the sum of its digits is a multiple of 9.

Proof:
Let N be a four digit number such that $N = abcd$.

$N = a \bullet 1000 + b \bullet 100 + c \bullet 10 + d$ ______________________________

$N = a(999+1) + b(99+1) + c(9+1) + d$ ______________________________

$N = 999a + a + 99b + b + 9c + c + d$ ______________________________

$N = 999a + 99b + 9c + a + b + c + d$ ______________________________

$N = 9 \bullet 111a + 9 \bullet 11b + 9 \bullet 1c + (a+b+c+d)$ ______________________________

Because it has 9 as a factor, $9(111a) + 9(11b) + 9(1c)$ is divisible by 9.
$\therefore N$ would only be divisible by 9 if $a+b+c+d$, the remainder, is divisible by 9.
And, if $a+b+c+d$ is divisible by 9, then N is divisible by 9.

Ex 4) Divisibility rule for 8: A number is divisible by 8 *iff* the last three digits form a number that is a multiple of 8.

Proof:
Let N be a four digit number such that $N = abcd$.

$N = a \bullet 1000 + b \bullet 100 + c \bullet 10 + d$ ______________________________

$N = a \bullet 125 \bullet 8 + b \bullet 100 + c \bullet 10 + d$ ______________________________

$N = 8 \bullet 125a + b \bullet 100 + c \bullet 10 + d$ ______________________________

Because it has 8 as a factor, $8(125a)$ is divisible by 8.
In other words, ALL thousands are divisible by 8.
$\therefore N$ is only divisible by 8 if $100b + 10c + d$, the number formed by the last three digits, is divisible by 8.
And, if $100b + 10c + d$ is divisible by 8, then N is divisible by 8.